

Derivation Hom-Lie 2-algebras and non-abelian extensions of Hom-Lie algebras *

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Abstract

In this paper, we introduce the notion of a derivation of a Hom-Lie algebra and construct the corresponding strict Hom-Lie 2-algebra, which is called the derivation Hom-Lie 2-algebra. As applications, we study non-abelian extensions of Hom-Lie algebras. We show that isomorphism classes of diagonal non-abelian extensions of a Hom-Lie algebra \mathfrak{g} by a Hom-Lie algebra \mathfrak{h} are in one-to-one correspondence with homotopy classes of morphisms from \mathfrak{g} to the derivation Hom-Lie 2-algebra $\text{DER}(\mathfrak{h})$.

1 Introduction

The notion of a Hom-Lie algebra was introduced by Hartwig, Larsson, and Silvestrov in [9] as part of a study of deformations of the Witt and the Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity. Some q -deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra [9, 11]. Because of close relation to discrete and deformed vector fields and differential calculus [9, 13, 14], more people pay special attention to this algebraic structure. In particular, representations and deformations of Hom-Lie algebras were studied in [3, 17, 19]; Extensions of Hom-Lie algebras were studied in [6, 13]. Geometric generalization of Hom-Lie algebras was given in [16]; Quantization of Hom-Lie algebras was studied in [22] and integration of Hom-Lie algebras was studied in [15].

Now higher categorical structures are very important due to connections with string theory [5]. One way to provide higher categorical structures is by categorifying existing mathematical concepts. A Lie 2-algebra is the categorification of a Lie algebra [4]. The Jacobi identity in a Lie 2-algebra is replaced by a natural transformation, called the Jacobiator, which also satisfies a coherence law of its own. A very important example is the derivation Lie 2-algebra $\text{DER}(\mathfrak{k}) = (\mathfrak{k} \xrightarrow{\text{ad}} \text{Der}(\mathfrak{k}), l_2)$ associated to a Lie algebra \mathfrak{k} , where l_2 is given by

$$l_2(D_1, D_2) = D_1 \circ D_2 - D_2 \circ D_1, \quad l_2(D, u) = -l_2(u, D) = D(u), \quad \forall D, D_1, D_2 \in \text{Der}(\mathfrak{k}), u \in \mathfrak{k}.$$

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In [18], a non-abelian extension of a Lie algebra \mathfrak{g} by a Lie algebra \mathfrak{h} is explained by a Lie 2-algebra morphism from \mathfrak{g} to the derivation Lie 2-algebra $\text{DER}(\mathfrak{h})$. The notion of a Hom-Lie 2-algebra, which is the categorification of a Hom-Lie algebra, was given in [20].

In this paper, we give the notion of a derivation of a Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_g)$. The set of derivations $\text{Der}(\mathfrak{g})$ is a Hom-Lie subalgebra of the Hom-Lie algebra $(\mathfrak{gl}(\mathfrak{g}), [\cdot, \cdot]_{\phi_g}, \text{Ad}_{\phi_g})$ which was given in [21]. We prove that the set of outer derivation $\text{Out}(\mathfrak{g})$ is exactly the first cohomology group $\mathcal{H}^1(\mathfrak{g}, \text{ad})$ of the Hom-Lie algebra \mathfrak{g} with the coefficient in the adjoint representation. Then we construct the derivation Hom-Lie 2-algebra $\text{DER}(\mathfrak{g})$. As applications, we study non-abelian extensions of Hom-Lie algebras. Parallel to the case of Lie algebras, we characterize a diagonal non-abelian extension of a Hom-Lie algebra \mathfrak{g} by a Hom-Lie algebra \mathfrak{h} using a Hom-Lie 2-algebra morphism from \mathfrak{g} to the derivation Hom-Lie 2-algebra $\text{DER}(\mathfrak{h})$.

The paper is organized as follows. In Section 2, we recall some basic notions of Hom-Lie algebras, representations of Hom-Lie algebras and their cohomologies, Hom-Lie 2-algebras and morphisms between Hom-Lie 2-algebras. In Section 3, we give the notion of a derivation of a Hom-Lie algebra and construct the associated derivation Hom-Lie 2-algebra. In Section 4, we study diagonal non-abelian extensions of Hom-Lie algebras and use Hom-Lie 2-algebra morphisms to characterize them. In Section 5, we give a discussion on some other ways to characterize diagonal non-abelian extensions of Hom-Lie algebras.

2 Preliminaries

In this section, we recall some basic notions of Hom-Lie algebras, representations of Hom-Lie algebras and their cohomologies, Hom-Lie 2-algebras and morphisms between Hom-Lie 2-algebras. Moreover, we give the notion of a 2-morphism between two morphisms of Hom-Lie 2-algebras.

2.1 Hom-Lie algebras and their representations

Definition 2.1. (1) A (multiplicative) Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_g)$ consisting of a vector space \mathfrak{g} , a skew-symmetric bilinear map (bracket) $[\cdot, \cdot]_{\mathfrak{g}} : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ and a linear map $\phi_g : \mathfrak{g} \rightarrow \mathfrak{g}$ preserving the bracket, such that the following Hom-Jacobi identity with respect to ϕ_g is satisfied:

$$[\phi_g(x), [y, z]_{\mathfrak{g}}]_{\mathfrak{g}} + [\phi_g(y), [z, x]_{\mathfrak{g}}]_{\mathfrak{g}} + [\phi_g(z), [x, y]_{\mathfrak{g}}]_{\mathfrak{g}} = 0. \quad (1)$$

(2) A Hom-Lie algebra is called a regular Hom-Lie algebra if ϕ_g is an algebra automorphism.

In the sequel, we always assume that ϕ_g is an algebra automorphism.

Definition 2.2. A morphism of Hom-Lie algebras $f : (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_g) \rightarrow (\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \phi_h)$ is a linear map $f : \mathfrak{g} \rightarrow \mathfrak{h}$ such that

$$f[x, y]_{\mathfrak{g}} = [f(x), f(y)]_{\mathfrak{h}}, \quad \forall x, y \in \mathfrak{g}, \quad (2)$$

$$f \circ \phi_g = \phi_h \circ f. \quad (3)$$

Definition 2.3. A representation of a Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_g)$ on a vector space V with respect to $\beta \in \mathfrak{gl}(V)$ is a linear map $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ such that for all $x, y \in \mathfrak{g}$, the following equalities are satisfied:

$$\rho(\phi_g(x)) \circ \beta = \beta \circ \rho(x), \quad (4)$$

$$\rho([x, y]_{\mathfrak{g}}) \circ \beta = \rho(\phi_g(x)) \circ \rho(y) - \rho(\phi_g(y)) \circ \rho(x). \quad (5)$$

We denote a representation by (ρ, V, β) . For all $x \in \mathfrak{g}$, we define $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$\text{ad}_x(y) = [x, y]_{\mathfrak{g}}, \quad \forall y \in \mathfrak{g}. \quad (6)$$

Then $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ is a representation of the Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ on \mathfrak{g} with respect to $\phi_{\mathfrak{g}}$, which is called the **adjoint representation**.

Let (ρ, V, β) be a representation. The cohomology of the Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ with the coefficient in V is the cohomology of the cochain complex $C^k(\mathfrak{g}, V) = \text{Hom}(\wedge^k \mathfrak{g}, V)$ with the coboundary operator $d : C^k(\mathfrak{g}, V) \rightarrow C^{k+1}(\mathfrak{g}, V)$ defined by

$$\begin{aligned} (df)(x_1, \dots, x_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \rho(x_i) (f(\phi_{\mathfrak{g}}^{-1} x_1, \dots, \widehat{\phi_{\mathfrak{g}}^{-1} x_i}, \dots, \phi_{\mathfrak{g}}^{-1} x_{k+1})) \\ &+ \sum_{i < j} (-1)^{i+j} \beta f([\phi_{\mathfrak{g}}^{-2} x_i, \phi_{\mathfrak{g}}^{-2} x_j]_{\mathfrak{g}}, \phi_{\mathfrak{g}}^{-1} x_1, \dots, \widehat{\phi_{\mathfrak{g}}^{-1} x_i}, \dots, \widehat{\phi_{\mathfrak{g}}^{-1} x_j}, \dots, \phi_{\mathfrak{g}}^{-1} x_{k+1}). \end{aligned}$$

The fact that $d^2 = 0$ is proved in [7]. Denote by $\mathcal{Z}^k(\mathfrak{g}; \rho)$ and $\mathcal{B}^k(\mathfrak{g}; \rho)$ the sets of k -cocycles and k -coboundaries respectively. We define the k -th cohomology group $\mathcal{H}^k(\mathfrak{g}; \rho)$ to be $\mathcal{Z}^k(\mathfrak{g}; \rho) / \mathcal{B}^k(\mathfrak{g}; \rho)$.

Let $(\text{ad}, \phi_{\mathfrak{g}})$ be the adjoint representation. For any 0-hom-cochain $x \in \mathfrak{g} = C^0(\mathfrak{g}, \mathfrak{g})$, we have $(dx)(y) = [y, x]_{\mathfrak{g}}$, for all $y \in \mathfrak{g}$. Thus, we have $dx = 0$ if and only if $x \in \text{Cen}(\mathfrak{g})$, where $\text{Cen}(\mathfrak{g})$ denotes the center of \mathfrak{g} . Therefore, we have

$$\mathcal{H}^0(\mathfrak{g}, \text{ad}) = \mathcal{Z}^0(\mathfrak{g}, \text{ad}) = \text{Cen}(\mathfrak{g}).$$

We will analyze the first cohomology group after we introduce the notion of a derivation of a Hom-Lie algebra.

Let V be a vector space, and $\beta \in GL(V)$. Define a skew-symmetric bilinear bracket operation $[\cdot, \cdot]_{\beta} : \wedge^2 \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)$ by

$$[A, B]_{\beta} = \beta \circ A \circ \beta^{-1} \circ B \circ \beta^{-1} - \beta \circ B \circ \beta^{-1} \circ A \circ \beta^{-1}, \quad \forall A, B \in \mathfrak{gl}(V). \quad (7)$$

Denote by $\text{Ad}_{\beta} : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)$ the adjoint action on $\mathfrak{gl}(V)$, i.e.

$$\text{Ad}_{\beta}(A) = \beta \circ A \circ \beta^{-1}. \quad (8)$$

Proposition 2.4. ([21, Proposition 4.1]) *With the above notations, $(\mathfrak{gl}(V), [\cdot, \cdot]_{\beta}, \text{Ad}_{\beta})$ is a regular Hom-Lie algebra.*

This Hom-Lie algebra plays an important role in the representation theory of Hom-Lie algebras. See [21] for more details.

2.2 Hom-Lie 2-algebras

Definition 2.5. ([20, Definition 3.6]) *A Hom-Lie 2-algebra \mathcal{V} consists of the following data:*

- a complex of vector spaces $V_1 \xrightarrow{d} V_0$,
- bilinear maps $l_2 : V_i \times V_j \rightarrow V_{i+j}$, where $0 \leq i + j \leq 1$,
- two linear transformations $\phi_0 \in \mathfrak{gl}(V_0), \phi_1 \in \mathfrak{gl}(V_1)$ satisfying $\phi_0 \circ d = d \circ \phi_1$,
- a skew-symmetric trilinear map $l_3 : V_0 \times V_0 \times V_0 \rightarrow V_1$ satisfying $l_3 \circ \phi_0^{\otimes 3} = \phi_1 \circ l_3$, such that for all $w, x, y, z \in V_0$ and $m, n \in V_1$, the following equalities are satisfied:

- (a) $l_2(x, y) = -l_2(y, x), \quad l_2(x, m) = -l_2(m, x),$
- (b) $dl_2(x, m) = l_2(x, dm), \quad l_2(dm, n) = l_2(m, dn),$
- (c) $\phi_0(l_2(x, y)) = l_2(\phi_0(x), \phi_0(y)), \quad \phi_1(l_2(x, m)) = l_2(\phi_0(x), \phi_1(m)),$
- (d) $dl_3(x, y, z) = l_2(\phi_0(x), l_2(y, z)) + l_2(\phi_0(y), l_2(z, x)) + l_2(\phi_0(z), l_2(x, y)),$
- (e) $l_3(x, y, dm) = l_2(\phi_0(x), l_2(y, m)) + l_2(\phi_0(y), l_2(m, x)) + l_2(\phi_1(m), l_2(x, y)),$
- (f) $l_3(l_2(w, x), \phi_0(y), \phi_0(z)) + l_2(l_3(w, x, z), \phi_0^2(y)) + l_3(\phi_0(w), l_2(x, z), \phi_0(y))$
 $+ l_3(l_2(w, z), \phi_0(x), \phi_0(y)) = l_2(l_3(w, x, y), \phi_0^2(z)) + l_3(l_2(w, y), \phi_0(x), \phi_0(z))$
 $+ l_3(\phi_0(w), l_2(x, y), \phi_0(z)) + l_2(\phi_0^2(w), l_3(x, y, z)) + l_2(l_3(w, y, z), \phi_0^2(x)) + l_3(\phi_0(w), l_2(y, z), \phi_0(x)).$

We usually denote a Hom-Lie 2-algebra by $(V_1, V_0, d, l_2, l_3, \phi_0, \phi_1)$ or simply by \mathcal{V} . A Hom-Lie 2-algebra is called strict if $l_3 = 0$. If ϕ_0 and ϕ_1 are identity maps, we obtain the notion of a Lie 2-algebra [4].

Definition 2.6. Let \mathcal{V} and \mathcal{V}' be Hom-Lie 2-algebras, a morphism f from \mathcal{V} to \mathcal{V}' consists of:

- a chain map $f : \mathcal{V} \rightarrow \mathcal{V}'$, which consists of linear maps $f_0 : V_0 \rightarrow V'_0$ and $f_1 : V_1 \rightarrow V'_1$ satisfying

$$f_0 \circ d = d' \circ f_1,$$

and

$$f_0 \circ \phi_0 = \phi'_0 \circ f_0, \quad f_1 \circ \phi_1 = \phi'_1 \circ f_1.$$

- a skew-symmetric bilinear map $f_2 : V_0 \times V_0 \rightarrow V'_1$ satisfying $f_2(\phi_0(x), \phi_0(y)) = \phi'_1 f_2(x, y)$ such that for all $x, y, z \in V_0$ and $m, n \in V_1$, we have
- $df_2(x, y) = f_0(l_2(x, y)) - l'_2(f_0(x), f_0(y)),$
- $f_2(x, dm) = f_1(l_2(x, m)) - l'_2(f_0(x), f_1(m)),$
- $l'_2(f_0(\phi_0(x)), f_2(y, z)) + l'_2(f_0(\phi_0(y)), f_2(z, x)) + l'_2(f_0(\phi_0(z)), f_2(x, y)) + l'_3(f_0(x), f_0(y), f_0(z))$
 $= f_2(l_2(x, y), \phi_0(z)) + f_2(l_2(y, z), \phi_0(x)) + f_2(l_2(z, x), \phi_0(y)) + f_1(l_3(x, y, z)).$

Definition 2.7. (1) Let \mathcal{V} and \mathcal{V}' be Hom-Lie 2-algebras and let $f, g : \mathcal{V} \rightarrow \mathcal{V}'$ be morphisms from \mathcal{V} to \mathcal{V}' . A 2-morphism τ from f to g is a chain homotopy $\tau : f \Rightarrow g$ such that the following equations hold for all $x, y \in V_0$:

$$\begin{aligned} \phi'_1(\tau(x)) &= \tau(\phi_0(x)), \\ f_2(x, y) - g_2(x, y) &= l'_2(\tau(x), f_0(y)) + l'_2(g_0(x), \tau(y)) - \tau(l_2(x, y)). \end{aligned}$$

- (2) Two morphisms between Hom-Lie 2-algebras are called homotopic if there is a 2-morphism between them.

3 Derivations of a Hom-Lie algebra and the associated derivation Hom-Lie 2-algebra

In this section, we introduce the notion of a derivation of a Hom-Lie 2-algebra and give its cohomological characterization. Then we construct a strict Hom-Lie 2-algebra using derivations of a Hom-Lie algebra, which is called the derivation Hom-Lie 2-algebra.

Definition 3.1. A linear map $D : \mathfrak{g} \rightarrow \mathfrak{g}$ is called a derivation of a Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ if

$$D[x, y]_{\mathfrak{g}} = [\phi_{\mathfrak{g}}(x), (\text{Ad}_{\phi_{\mathfrak{g}}^{-1}} D)(y)]_{\mathfrak{g}} + [(\text{Ad}_{\phi_{\mathfrak{g}}^{-1}} D)(x), \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}. \quad (9)$$

Denote by $\text{Der}(\mathfrak{g})$ the set of derivations of the Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$.

Remark 3.2. The notion of an α -derivation was given in [19] under an extra condition $D \circ \phi_{\mathfrak{g}} = \phi_{\mathfrak{g}} \circ D$. Under this condition, $\text{Ad}_{\phi_{\mathfrak{g}}} D = D$, and it follows that our derivation is the same as the one given in [19]. So our definition of a derivation of a Hom-Lie algebra is more general than the one given in [19]. We will show that the set of derivations is a Hom-Lie subalgebra of the Hom-Lie algebra $(\mathfrak{gl}(\mathfrak{g}), [\cdot, \cdot]_{\phi_{\mathfrak{g}}}, \text{Ad}_{\phi_{\mathfrak{g}}})$ given in Proposition 2.4 and the first cohomology group of a Hom-Lie algebra with the coefficient in the adjoint representation is exactly the set of outer derivations. These facts show that our definition of a derivation of a Hom-Lie algebra is more meaningful. More importantly, we can construct a Hom-Lie 2-algebra using the derivations, by which we characterize nonabelian extensions of Hom-Lie algebras.

Lemma 3.3. For all $D \in \text{Der}(\mathfrak{g})$, we have $\text{Ad}_{\phi_{\mathfrak{g}}} D \in \text{Der}(\mathfrak{g})$.

Proof. By (8) and (9), we have

$$\begin{aligned} (\text{Ad}_{\phi_{\mathfrak{g}}} D)[x, y]_{\mathfrak{g}} &= (\phi_{\mathfrak{g}} \circ D \circ \phi_{\mathfrak{g}}^{-1})[x, y]_{\mathfrak{g}} = (\phi_{\mathfrak{g}} \circ D)(\phi_{\mathfrak{g}}^{-1}x, \phi_{\mathfrak{g}}^{-1}y)_{\mathfrak{g}} \\ &= \phi_{\mathfrak{g}}[x, \phi_{\mathfrak{g}}^{-1}D(y)]_{\mathfrak{g}} + \phi_{\mathfrak{g}}[\phi_{\mathfrak{g}}^{-1}D(x), y]_{\mathfrak{g}} \\ &= [\phi_{\mathfrak{g}}x, (\text{Ad}_{\phi_{\mathfrak{g}}^{-1}}(\text{Ad}_{\phi_{\mathfrak{g}}} D))(y)]_{\mathfrak{g}} + [(\text{Ad}_{\phi_{\mathfrak{g}}^{-1}}(\text{Ad}_{\phi_{\mathfrak{g}}} D))(x), \phi_{\mathfrak{g}}y]_{\mathfrak{g}}. \end{aligned}$$

Thus, $\text{Ad}_{\phi_{\mathfrak{g}}} D \in \text{Der}(\mathfrak{g})$. ■

Consider the Hom-Lie bracket $[\cdot, \cdot]_{\phi_{\mathfrak{g}}}$ on $\mathfrak{gl}(\mathfrak{g})$ defined by (7), we have

Lemma 3.4. For all $D, D' \in \text{Der}(\mathfrak{g})$, we have $[D, D']_{\phi_{\mathfrak{g}}} \in \text{Der}(\mathfrak{g})$.

Proof. For all $x, y \in \mathfrak{g}$, by (7) and (9) we have

$$\begin{aligned} [D, D']_{\phi_{\mathfrak{g}}}([x, y]_{\mathfrak{g}}) &= (\phi_{\mathfrak{g}} \circ D \circ \phi_{\mathfrak{g}}^{-1} \circ D' \circ \phi_{\mathfrak{g}}^{-1} - \phi_{\mathfrak{g}} \circ D' \circ \phi_{\mathfrak{g}}^{-1} \circ D \circ \phi_{\mathfrak{g}}^{-1})([x, y]_{\mathfrak{g}}) \\ &= (\phi_{\mathfrak{g}} \circ D \circ \phi_{\mathfrak{g}}^{-1} \circ D')([\phi_{\mathfrak{g}}^{-1}x, \phi_{\mathfrak{g}}^{-1}y]_{\mathfrak{g}}) - (\phi_{\mathfrak{g}} \circ D' \circ \phi_{\mathfrak{g}}^{-1} \circ D)([\phi_{\mathfrak{g}}^{-1}x, \phi_{\mathfrak{g}}^{-1}y]_{\mathfrak{g}}) \\ &= (\phi_{\mathfrak{g}} \circ D \circ \phi_{\mathfrak{g}}^{-1})([x, \phi_{\mathfrak{g}}^{-1}(D'(y))]_{\mathfrak{g}}) + (\phi_{\mathfrak{g}} \circ D \circ \phi_{\mathfrak{g}}^{-1})([\phi_{\mathfrak{g}}^{-1}(D'(x)), y]_{\mathfrak{g}}) \\ &\quad - (\phi_{\mathfrak{g}} \circ D' \circ \phi_{\mathfrak{g}}^{-1})([x, \phi_{\mathfrak{g}}^{-1}(D(y))]_{\mathfrak{g}}) - (\phi_{\mathfrak{g}} \circ D' \circ \phi_{\mathfrak{g}}^{-1})([\phi_{\mathfrak{g}}^{-1}(D(x)), y]_{\mathfrak{g}}) \end{aligned}$$

$$\begin{aligned}
&= (\phi_{\mathfrak{g}} \circ D)([\phi_{\mathfrak{g}}^{-1}x, \phi_{\mathfrak{g}}^{-2}(D'(y))]_{\mathfrak{g}}) + (\phi_{\mathfrak{g}} \circ D)([\phi_{\mathfrak{g}}^{-2}(D'(x)), \phi_{\mathfrak{g}}^{-1}y]_{\mathfrak{g}}) \\
&\quad - (\phi_{\mathfrak{g}} \circ D')([\phi_{\mathfrak{g}}^{-1}(x), \phi_{\mathfrak{g}}^{-2}(D(y))]_{\mathfrak{g}}) - (\phi_{\mathfrak{g}} \circ D')([\phi_{\mathfrak{g}}^{-2}(D(x)), \phi_{\mathfrak{g}}^{-1}(y)]_{\mathfrak{g}}) \\
&= \phi_{\mathfrak{g}}[x, \phi_{\mathfrak{g}}^{-1}(D(\phi_{\mathfrak{g}}^{-1}(D'(y))))]_{\mathfrak{g}} + \phi_{\mathfrak{g}}[\phi_{\mathfrak{g}}^{-1}(D(x)), \phi_{\mathfrak{g}}^{-1}(D'(y))]_{\mathfrak{g}} \\
&\quad + \phi_{\mathfrak{g}}[\phi_{\mathfrak{g}}^{-1}(D'(x)), \phi_{\mathfrak{g}}^{-1}(D(y))]_{\mathfrak{g}} + \phi_{\mathfrak{g}}[\phi_{\mathfrak{g}}^{-1}(D(\phi_{\mathfrak{g}}^{-1}(D'(x)))), y]_{\mathfrak{g}} \\
&\quad - \phi_{\mathfrak{g}}[x, \phi_{\mathfrak{g}}^{-1}(D'(\phi_{\mathfrak{g}}^{-1}(D(y))))]_{\mathfrak{g}} - \phi_{\mathfrak{g}}[\phi_{\mathfrak{g}}^{-1}(D'(x)), \phi_{\mathfrak{g}}^{-1}(D(y))]_{\mathfrak{g}} \\
&\quad - \phi_{\mathfrak{g}}[\phi_{\mathfrak{g}}^{-1}(D'(\phi_{\mathfrak{g}}^{-1}(D(x)))), y]_{\mathfrak{g}} - \phi_{\mathfrak{g}}[\phi_{\mathfrak{g}}^{-1}(D(x)), \phi_{\mathfrak{g}}^{-1}(D'(y))]_{\mathfrak{g}} \\
&= [\phi_{\mathfrak{g}}(x), (D \circ \phi_{\mathfrak{g}}^{-1} \circ D' - D' \circ \phi_{\mathfrak{g}}^{-1} \circ D)(y)]_{\mathfrak{g}} \\
&\quad + [(D \circ \phi_{\mathfrak{g}}^{-1} \circ D' - D' \circ \phi_{\mathfrak{g}}^{-1} \circ D)(x), \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}} \\
&= [\phi_{\mathfrak{g}}(x), (\text{Ad}_{\phi_{\mathfrak{g}}}^{-1}[D, D']_{\phi_{\mathfrak{g}}})(y)]_{\mathfrak{g}} + [(\text{Ad}_{\phi_{\mathfrak{g}}}^{-1}[D, D']_{\phi_{\mathfrak{g}}})(x), \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}.
\end{aligned}$$

Therefore, we have $[D, D']_{\phi_{\mathfrak{g}}} \in \text{Der}(\mathfrak{g})$. ■

Proposition 3.5. *With the above notations, $(\text{Der}(\mathfrak{g}), [\cdot, \cdot]_{\mathfrak{g}}, \text{Ad}_{\phi_{\mathfrak{g}}})$ is a Hom-Lie algebra, which is a sub-algebra of the Hom-Lie algebra $(\mathfrak{gl}(\mathfrak{g}), [\cdot, \cdot]_{\phi_{\mathfrak{g}}}, \text{Ad}_{\phi_{\mathfrak{g}}})$ given in Proposition 2.4.*

Proof. By Lemma 3.3 and 3.4, $(\text{Der}(\mathfrak{g}), [\cdot, \cdot]_{\mathfrak{g}}, \text{Ad}_{\phi_{\mathfrak{g}}})$ is a Hom-Lie subalgebra of the Hom-Lie algebra $(\mathfrak{gl}(\mathfrak{g}), [\cdot, \cdot]_{\phi_{\mathfrak{g}}}, \text{Ad}_{\phi_{\mathfrak{g}}})$. ■

For all $x \in \mathfrak{g}$, ad_x is a derivation of the Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$, which we call an **inner derivation**. This follows from

$$\begin{aligned}
\text{ad}_x[y, z]_{\mathfrak{g}} = [x, [y, z]_{\mathfrak{g}}]_{\mathfrak{g}} &= [\phi_{\mathfrak{g}}(\phi_{\mathfrak{g}}^{-1}x), [y, z]_{\mathfrak{g}}]_{\mathfrak{g}} \\
&= [[\phi_{\mathfrak{g}}^{-1}x, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}} + [\phi_{\mathfrak{g}}(y), [\phi_{\mathfrak{g}}^{-1}x, z]_{\mathfrak{g}}]_{\mathfrak{g}} \\
&= [\phi_{\mathfrak{g}}^{-1}[x, \phi_{\mathfrak{g}}(y)]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}} + [\phi_{\mathfrak{g}}(y), \phi_{\mathfrak{g}}^{-1}[x, \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}}]_{\mathfrak{g}} \\
&= [(\text{Ad}_{\phi_{\mathfrak{g}}^{-1}}\text{ad}_x)(y), \phi_{\mathfrak{g}}(z)]_{\mathfrak{g}} + [\phi_{\mathfrak{g}}(y), (\text{Ad}_{\phi_{\mathfrak{g}}^{-1}}\text{ad}_x)(z)]_{\mathfrak{g}}.
\end{aligned}$$

Denote by $\text{Inn}(\mathfrak{g})$ the set of inner derivations of the Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$, i.e.

$$\text{Inn}(\mathfrak{g}) = \{\text{ad}_x \mid x \in \mathfrak{g}\}. \quad (10)$$

Lemma 3.6. *Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ be a Hom-Lie algebra. For all $x \in \mathfrak{g}$ and $D \in \text{Der}(\mathfrak{g})$, we have*

$$\text{Ad}_{\phi_{\mathfrak{g}}}\text{ad}_x = \text{ad}_{\phi_{\mathfrak{g}}(x)}, \quad [D, \text{ad}_x]_{\phi_{\mathfrak{g}}} = \text{ad}_{D(x)}.$$

Therefore, $\text{Inn}(\mathfrak{g})$ is an ideal of the Hom-Lie algebra $(\text{Der}(\mathfrak{g}), [\cdot, \cdot]_{\mathfrak{g}}, \text{Ad}_{\phi_{\mathfrak{g}}})$.

Proof. For all $x, y \in \mathfrak{g}$, by (8), we have

$$\begin{aligned}
(\text{Ad}_{\phi_{\mathfrak{g}}}\text{ad}_x)(y) &= (\phi_{\mathfrak{g}} \circ \text{ad}_x \circ \phi_{\mathfrak{g}}^{-1})(y) = (\phi_{\mathfrak{g}} \circ \text{ad}_x)(\phi_{\mathfrak{g}}^{-1}(y)) = \phi_{\mathfrak{g}}[x, \phi_{\mathfrak{g}}^{-1}(y)]_{\mathfrak{g}} = [\phi_{\mathfrak{g}}(x), y]_{\mathfrak{g}} \\
&= \text{ad}_{\phi_{\mathfrak{g}}(x)}(y).
\end{aligned}$$

By (9), we have

$$\begin{aligned}
[D, \text{ad}_x]_{\phi_{\mathfrak{g}}}(y) &= (\phi_{\mathfrak{g}} \circ D \circ \phi_{\mathfrak{g}}^{-1} \circ \text{ad}_x \circ \phi_{\mathfrak{g}}^{-1})(y) - (\phi_{\mathfrak{g}} \circ \text{ad}_x \circ \phi_{\mathfrak{g}}^{-1} \circ D \circ \phi_{\mathfrak{g}}^{-1})(y) \\
&= (\phi_{\mathfrak{g}} \circ D)(\phi_{\mathfrak{g}}^{-1}[x, \phi_{\mathfrak{g}}^{-1}(y)]_{\mathfrak{g}}) - (\phi_{\mathfrak{g}} \circ \text{ad}_x \circ \phi_{\mathfrak{g}}^{-1})(D(\phi_{\mathfrak{g}}^{-1}(y))) \\
&= \phi_{\mathfrak{g}}(D[\phi_{\mathfrak{g}}^{-1}x, \phi_{\mathfrak{g}}^{-2}(y)]_{\mathfrak{g}}) - \phi_{\mathfrak{g}}[x, \phi_{\mathfrak{g}}^{-1}D(\phi_{\mathfrak{g}}^{-1}(y))]_{\mathfrak{g}} \\
&= \phi_{\mathfrak{g}}\left([x, (\text{Ad}_{\phi_{\mathfrak{g}}^{-1}}D)(\phi_{\mathfrak{g}}^{-2}(y))]_{\mathfrak{g}} + [(\text{Ad}_{\phi_{\mathfrak{g}}^{-1}}D)(\phi_{\mathfrak{g}}^{-1}(x)), \phi_{\mathfrak{g}}^{-1}(y)]_{\mathfrak{g}}\right) - [\phi_{\mathfrak{g}}(x), D(\phi_{\mathfrak{g}}^{-1}(y))]_{\mathfrak{g}} \\
&= [D(x), y]_{\mathfrak{g}} \\
&= \text{ad}_{D(x)}(y).
\end{aligned}$$

Therefore, we have $\text{Ad}_{\phi_g} \text{ad}_x = \text{ad}_{\phi_g(x)}$ and $[D, \text{ad}_x]_{\phi_g} = \text{ad}_{D(x)}$. The proof is finished. ■

Denote by $\text{Out}(\mathfrak{g})$ the set of out derivations of the Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_g)$, i.e.

$$\text{Out}(\mathfrak{g}) = \text{Der}(\mathfrak{g}) / \text{Inn}(\mathfrak{g}). \quad (11)$$

Proposition 3.7. *Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_g)$ be a Hom-Lie algebra. We have*

$$\mathcal{H}^1(\mathfrak{g}, \text{ad}) = \text{Out}(\mathfrak{g}).$$

Proof. For any $f \in C^1(\mathfrak{g}, \mathfrak{g})$, we have

$$(df)(x_1, x_2) = [x_1, f(\phi_g^{-1}x_2)]_{\mathfrak{g}} - [x_2, f(\phi_g^{-1}x_1)]_{\mathfrak{g}} - \phi_g f([\phi_g^{-2}x_1, \phi_g^{-2}x_2]_{\mathfrak{g}}).$$

Therefore, the set of 1-cocycles $\mathcal{Z}^1(\mathfrak{g}, \text{ad})$ is given by

$$\begin{aligned} f([\phi_g^{-2}x_1, \phi_g^{-2}x_2]_{\mathfrak{g}}) &= [\phi_g^{-1}x_1, \phi_g^{-1}f(\phi_g^{-1}x_2)]_{\mathfrak{g}} + [\phi_g^{-1}f(\phi_g^{-1}x_1), \phi_g^{-1}x_2]_{\mathfrak{g}} \\ &= [\phi_g(\phi_g^{-2}x_1), (\text{Ad}_{\phi_g^{-1}}f)(\phi_g^{-2}x_2)]_{\mathfrak{g}} + [(\text{Ad}_{\phi_g^{-1}}f)(\phi_g^{-2}x_1), \phi_g(\phi_g^{-2}x_2)]_{\mathfrak{g}}. \end{aligned}$$

Thus, we have $\mathcal{Z}^1(\mathfrak{g}, \text{ad}) = \text{Der}(\mathfrak{g})$.

Furthermore, the set of 1-coboundaries $\mathcal{B}^1(\mathfrak{g}, \text{ad})$ is given by

$$dx = [\cdot, x]_{\mathfrak{g}} = \text{ad}_{-x},$$

for some $x \in \mathfrak{g}$. Therefore, we have $\mathcal{B}^1(\mathfrak{g}, \text{ad}) = \text{Inn}(\mathfrak{g})$, which implies that $\mathcal{H}^1(\mathfrak{g}, \text{ad}) = \text{Out}(\mathfrak{g})$. ■

At the end of this section, we construct a strict Hom-Lie 2-algebra using derivations of a Hom-Lie algebra. We call this strict Hom-Lie 2-algebra the **derivation Hom-Lie 2-algebra**. It plays an important role in our later study of nonabelian extensions of Hom-Lie algebras.

Let $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \phi_h)$ be a Hom-Lie algebra. Consider the complex $\mathfrak{h} \xrightarrow{\text{ad}} \text{Der}(\mathfrak{h})$, where \mathfrak{h} is of degree 1 and $\text{Der}(\mathfrak{h})$ is of degree 0. Define l_2 by

$$l_2(D_1, D_2) = [D_1, D_2]_{\phi_h}, \quad \forall D_1, D_2 \in \text{Der}(\mathfrak{h}), \quad (12)$$

$$l_2(D, u) = -l_2(u, D) = D(u), \quad \forall D \in \text{Der}(\mathfrak{h}), \quad u \in \mathfrak{h}. \quad (13)$$

Theorem 3.8. *With the above notations, $(\mathfrak{h}, \text{Der}(\mathfrak{h}), d = \text{ad}, l_2, \phi_0 = \text{Ad}_{\phi_h}, \phi_1 = \phi_h)$ is a strict Hom-Lie 2-algebra.*

We denote this strict Hom-Lie 2-algebra by $\text{DER}(\mathfrak{h})$.

Proof. First we show that the condition $\phi_0 \circ d = d \circ \phi_1$ holds. It follows from

$$\phi_0(d(u))(v) = (\text{Ad}_{\phi_h}(\text{ad}_u))(v) = (\phi_h \circ \text{ad}_u \circ \phi_h^{-1})(v) = [\phi_h(u), v]_{\mathfrak{h}},$$

and

$$d(\phi_1(u))(v) = d(\phi_h(u))(v) = \text{ad}_{\phi_h(u)}(v) = [\phi_h(u), v]_{\mathfrak{h}}.$$

Condition (a) in Definition 2.5 holds obviously.

For all $D \in \text{Der}(\mathfrak{h})$ and $u \in \mathfrak{h}$, we have

$$d(l_2(D, u))(v) = d(D(u))(v) = \text{ad}_{D(u)}(v) = [D(u), v]_{\mathfrak{h}}.$$

By Lemma 3.6, we have

$$l_2(D, du)(v) = [D, \mathbf{ad}_u]_{\phi_{\mathfrak{h}}}(v) = [D(u), v]_{\mathfrak{h}} = d(l_2(D, u))(v). \quad (14)$$

For all $u, v \in \mathfrak{h}$, we have

$$l_2(du, v) = l_2(\mathbf{ad}_u, v) = [u, v]_{\mathfrak{h}} = -[v, u]_{\mathfrak{h}} = -l_2(\mathbf{ad}_v, u) = l_2(u, \mathbf{ad}_v) = l_2(u, dv). \quad (15)$$

By (14) and (15), we deduce that Condition (b) holds.

Conditions (c) and (d) follow from the fact that $(\text{Der}(\mathfrak{h}), [\cdot, \cdot]_{\mathfrak{h}}, \phi_0 = \text{Ad}_{\phi_{\mathfrak{h}}})$ is a Hom-Lie algebra. Condition (e) follows from

$$\begin{aligned} & l_2(\phi_0(D_1), l_2(D_2, u)) + l_2(\phi_0(D_2), l_2(u, D_1)) + l_2(\phi_1(u), l_2(D_1, D_2)) \\ &= l_2(\text{Ad}_{\phi_{\mathfrak{h}}}(D_1), D_2(u)) + l_2(\text{Ad}_{\phi_{\mathfrak{h}}}(D_2), -D_1(u)) + l_2(\phi_{\mathfrak{h}}(u), [D_1, D_2]_{\phi_{\mathfrak{h}}}) \\ &= (\phi_{\mathfrak{h}} \circ D_1 \circ \phi_{\mathfrak{h}}^{-1})(D_2(u)) - (\phi_{\mathfrak{h}} \circ D_2 \circ \phi_{\mathfrak{h}}^{-1})(D_1(u)) \\ &\quad + l_2(\phi_{\mathfrak{h}}(u), \phi_{\mathfrak{h}} \circ D_1 \circ \phi_{\mathfrak{h}}^{-1} \circ D_2 \circ \phi_{\mathfrak{h}}^{-1} - \phi_{\mathfrak{h}} \circ D_2 \circ \phi_{\mathfrak{h}}^{-1} \circ D_1 \circ \phi_{\mathfrak{h}}^{-1}) = 0. \end{aligned}$$

Since $l_3 = 0$, Condition (f) holds naturally. The proof is finished. ■

4 Non-abelian extensions of Hom-Lie algebras

In this section, we characterize diagonal non-abelian extensions of Hom-Lie algebras by Hom-Lie 2-algebra morphisms from a Hom-Lie algebra (viewed as a trivial Hom-Lie 2-algebra) to the derivation Hom-Lie 2-algebra constructed in the last section.

Definition 4.1. A non-abelian extension of a Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ by a Hom-Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \phi_{\mathfrak{h}})$ is a commutative diagram with rows being short exact sequence of Hom-Lie algebra morphisms:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{h} & \xrightarrow{\iota} & \hat{\mathfrak{g}} & \xrightarrow{p} & \mathfrak{g} \longrightarrow 0 \\ & & \phi_{\mathfrak{h}} \downarrow & & \phi_{\hat{\mathfrak{g}}} \downarrow & & \phi_{\mathfrak{g}} \downarrow \\ 0 & \longrightarrow & \mathfrak{h} & \xrightarrow{\iota} & \hat{\mathfrak{g}} & \xrightarrow{p} & \mathfrak{g} \longrightarrow 0, \end{array}$$

where $(\hat{\mathfrak{g}}, [\cdot, \cdot]_{\hat{\mathfrak{g}}}, \phi_{\hat{\mathfrak{g}}})$ is a Hom-Lie algebra.

We can regard \mathfrak{h} as a subspace of $\hat{\mathfrak{g}}$ and $\phi_{\hat{\mathfrak{g}}}|_{\mathfrak{h}} = \phi_{\mathfrak{h}}$. Thus, \mathfrak{h} is an invariant subspace of $\phi_{\hat{\mathfrak{g}}}$. We say that an extension is **diagonal** if $\hat{\mathfrak{g}}$ has an invariant subspace X of $\phi_{\hat{\mathfrak{g}}}$ such that $\mathfrak{h} \oplus X = \hat{\mathfrak{g}}$. In general, $\hat{\mathfrak{g}}$ does not always have an invariant subspace X of $\phi_{\hat{\mathfrak{g}}}$ such that $\mathfrak{h} \oplus X = \hat{\mathfrak{g}}$. For example, the matrix representation of $\phi_{\hat{\mathfrak{g}}}$ is a Jordan block. We only study diagonal non-abelian extensions in the sequel.

Definition 4.2. Two extensions of \mathfrak{g} by \mathfrak{h} , $(\hat{\mathfrak{g}}_1, [\cdot, \cdot]_{\hat{\mathfrak{g}}_1}, \phi_{\hat{\mathfrak{g}}_1})$ and $(\hat{\mathfrak{g}}_2, [\cdot, \cdot]_{\hat{\mathfrak{g}}_2}, \phi_{\hat{\mathfrak{g}}_2})$, are said to be isomorphic if there exists a Hom-Lie algebra morphism $\theta : \hat{\mathfrak{g}}_2 \rightarrow \hat{\mathfrak{g}}_1$ such that we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{h} & \xrightarrow{\iota_2} & \hat{\mathfrak{g}}_2 & \xrightarrow{p_2} & \mathfrak{g} \longrightarrow 0 \\ & & \parallel & & \theta \downarrow & & \parallel \\ 0 & \longrightarrow & \mathfrak{h} & \xrightarrow{\iota_1} & \hat{\mathfrak{g}}_1 & \xrightarrow{p_1} & \mathfrak{g} \longrightarrow 0. \end{array}$$

Proposition 4.3. *Let $(\hat{\mathfrak{g}}_1, [\cdot, \cdot]_{\hat{\mathfrak{g}}_1}, \phi_{\hat{\mathfrak{g}}_1})$ and $(\hat{\mathfrak{g}}_2, [\cdot, \cdot]_{\hat{\mathfrak{g}}_2}, \phi_{\hat{\mathfrak{g}}_2})$ be two isomorphic extensions of a Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ by a Hom-Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \phi_{\mathfrak{h}})$. Then $\hat{\mathfrak{g}}_1$ is a diagonal non-abelian extension if and only if $\hat{\mathfrak{g}}_2$ is also a diagonal non-abelian extension.*

Proof. Let $\hat{\mathfrak{g}}_2$ be a diagonal non-abelian extension. Then it has an invariant subspace X of $\phi_{\hat{\mathfrak{g}}_2}$ such that $\mathfrak{h} \oplus X = \hat{\mathfrak{g}}_2$. Since θ is a Hom-Lie algebra morphism, for all $u \in X$, we have

$$\phi_{\hat{\mathfrak{g}}_1}(\theta u) = \theta(\phi_{\hat{\mathfrak{g}}_2} u).$$

Therefore, $\theta(X)$ is an invariant subspace of $\phi_{\hat{\mathfrak{g}}_1}$. Moreover, we have $\mathfrak{h} \oplus \theta(X) = \hat{\mathfrak{g}}_1$. Thus, $\hat{\mathfrak{g}}_1$ is a diagonal non-abelian extension. ■

A section of an extension $\hat{\mathfrak{g}}$ of \mathfrak{g} by \mathfrak{h} is a linear map $s : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ such that $p \circ s = \text{Id}$.

Lemma 4.4. *A Hom-Lie algebra $(\hat{\mathfrak{g}}, [\cdot, \cdot]_{\hat{\mathfrak{g}}}, \phi_{\hat{\mathfrak{g}}})$ is a diagonal non-abelian extension of a Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ by a Hom-Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \phi_{\mathfrak{h}})$ if and only if there is a section $s : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ such that*

$$\phi_{\hat{\mathfrak{g}}} \circ s = s \circ \phi_{\mathfrak{g}}. \quad (16)$$

A section $s : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ of $\hat{\mathfrak{g}}$ is called **diagonal** if (16) is satisfied.

Proof. Let $\hat{\mathfrak{g}}$ be a diagonal non-abelian extension of \mathfrak{g} by \mathfrak{h} . Then $\hat{\mathfrak{g}}$ has an invariant subspace X of $\phi_{\hat{\mathfrak{g}}}$ such that $\mathfrak{h} \oplus X = \hat{\mathfrak{g}}$. By the exactness, we have $p|_X : X \rightarrow \mathfrak{g}$ is a linear isomorphism. Thus we have a section $s = p|_X^{-1} : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$, such that $s(\mathfrak{g}) = X$ and $p \circ s = \text{Id}$. Since p is a Hom-Lie algebra morphism, we have

$$p(\phi_{\hat{\mathfrak{g}}}(s(x)) - s(\phi_{\mathfrak{g}}(x))) = 0, \quad \forall x \in \mathfrak{g}.$$

Thus, we have $\phi_{\hat{\mathfrak{g}}}(s(x)) - s(\phi_{\mathfrak{g}}(x)) \in \mathfrak{h}$. Moreover, since X is an invariant subspace of $\phi_{\hat{\mathfrak{g}}}$, we have

$$\phi_{\hat{\mathfrak{g}}}(s(x)) - s(\phi_{\mathfrak{g}}(x)) \in X,$$

which implies that

$$\phi_{\hat{\mathfrak{g}}}(s(x)) - s(\phi_{\mathfrak{g}}(x)) \in \mathfrak{h} \cap X = \{0\}.$$

Therefore, $\phi_{\hat{\mathfrak{g}}}(s(x)) = s(\phi_{\mathfrak{g}}(x))$.

Conversely, let $s : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ be a section such that

$$\phi_{\hat{\mathfrak{g}}}(s(x)) = s(\phi_{\mathfrak{g}}(x)), \quad \forall x \in \mathfrak{g}.$$

Then $s(\mathfrak{g})$ is an invariant subspace of $\phi_{\hat{\mathfrak{g}}}$. By the exactness, we have $\mathfrak{h} \oplus s(\mathfrak{g}) = \hat{\mathfrak{g}}$. Hence, the extension is diagonal. ■

Let $(\hat{\mathfrak{g}}, [\cdot, \cdot]_{\hat{\mathfrak{g}}}, \phi_{\hat{\mathfrak{g}}})$ be a diagonal extension of a Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ by a Hom-Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \phi_{\mathfrak{h}})$ and $s : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ a diagonal section. Define linear maps $\omega : \mathfrak{g} \wedge \mathfrak{g} \rightarrow \mathfrak{h}$ and $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{h})$ respectively by

$$\omega(x, y) = [s(x), s(y)]_{\hat{\mathfrak{g}}} - s[x, y]_{\mathfrak{g}}, \quad \forall x, y \in \mathfrak{g}, \quad (17)$$

$$\rho_x(u) = [s(x), u]_{\hat{\mathfrak{g}}}, \quad \forall x \in \mathfrak{g}, u \in \mathfrak{h}. \quad (18)$$

Obviously, $\hat{\mathfrak{g}}$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{h}$ as vector spaces. Transfer the Hom-Lie algebra structure on $\hat{\mathfrak{g}}$ to that on $\mathfrak{g} \oplus \mathfrak{h}$, we obtain a Hom-Lie algebra $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\rho, \omega)}, \phi)$, where $[\cdot, \cdot]_{(\rho, \omega)}$ and ϕ are given by

$$[x + u, y + v]_{(\rho, \omega)} = [x, y]_{\mathfrak{g}} + \omega(x, y) + \rho_x(v) - \rho_y(u) + [u, v]_{\mathfrak{h}}, \quad (19)$$

$$\phi(x + u) = \phi_{\mathfrak{g}}(x) + \phi_{\mathfrak{h}}(u). \quad (20)$$

The following proposition gives the conditions on ρ and ω such that $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\rho, \omega)}, \phi)$ is a Hom-Lie algebra.

Proposition 4.5. *With the above notations, $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\rho, \omega)}, \phi)$ is a Hom-Lie algebra if and only if ρ and ω satisfy the following equalities:*

$$\phi_{\mathfrak{h}} \circ \omega = \omega \circ \phi_{\mathfrak{g}}^{\otimes 2}, \quad (21)$$

$$\phi_{\mathfrak{h}} \circ \rho_x = \rho_{\phi_{\mathfrak{g}}(x)} \circ \phi_{\mathfrak{h}}, \quad (22)$$

$$\rho_x([u, v]_{\mathfrak{h}}) = [\phi_{\mathfrak{h}}(u), (\text{Ad}_{\phi_{\mathfrak{h}}^{-1}} \rho_x)(v)]_{\mathfrak{h}} + [(\text{Ad}_{\phi_{\mathfrak{h}}^{-1}} \rho_x)(u), \phi_{\mathfrak{h}}(v)]_{\mathfrak{h}}, \quad (23)$$

$$[\rho_x, \rho_y]_{\phi_{\mathfrak{h}}} - \rho_{[x, y]_{\mathfrak{g}}} = \text{ad}_{\omega(x, y)}, \quad (24)$$

$$\rho_{\phi_{\mathfrak{g}}(x)}(\omega(y, z)) + c.p. = \omega([x, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)) + c.p.. \quad (25)$$

Proof. If $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\rho, \omega)}, \phi)$ is a Hom-Lie algebra. By

$$\phi([x + u, y + v]_{(\rho, \omega)}) = [\phi(x + u), \phi(y + v)]_{(\rho, \omega)},$$

we deduce that (21) and (22) holds. By (22) and

$$[[u, v]_{(\rho, \omega)}, \phi(x)]_{(\rho, \omega)} + [[v, x]_{(\rho, \omega)}, \phi(u)]_{(\rho, \omega)} + [[x, u]_{(\rho, \omega)}, \phi(v)]_{(\rho, \omega)} = 0,$$

we deduce that (23) holds. By (22) and

$$[[u, x]_{(\rho, \omega)}, \phi(y)]_{(\rho, \omega)} + [[x, y]_{(\rho, \omega)}, \phi(u)]_{(\rho, \omega)} + [[y, u]_{(\rho, \omega)}, \phi(x)]_{(\rho, \omega)} = 0,$$

we deduce that (24) holds. By

$$[[x, y]_{(\rho, \omega)}, \phi(z)]_{(\rho, \omega)} + [[y, z]_{(\rho, \omega)}, \phi(x)]_{(\rho, \omega)} + [[z, x]_{(\rho, \omega)}, \phi(y)]_{(\rho, \omega)} = 0,$$

we deduce that (25) holds.

Conversely, if (21)-(25) hold, it is straightforward to see that $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\rho, \omega)}, \phi)$ is a Hom-Lie algebra. The proof is finished. ■

Remark 4.6. *Note that (23) implies that $\rho_x \in \text{Der}(\mathfrak{h})$.*

Obviously, we have

Corollary 4.7. *If ρ and ω satisfy Eqs. (21)-(25), the Hom-Lie algebra $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\rho, \omega)}, \phi)$ is a diagonal non-abelian extension of \mathfrak{g} by \mathfrak{h} .*

For any diagonal nonabelian extensions, by choosing a diagonal section, it is isomorphic to $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\rho, \omega)}, \phi)$. Therefore, we only consider diagonal nonabelian extensions of the form $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\rho, \omega)}, \phi)$ in the sequel.

Theorem 4.8. Let $\hat{\mathfrak{g}} = (\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\rho, \omega)}, \phi)$ be a diagonal non-abelian extension of \mathfrak{g} by \mathfrak{h} . Then, (ρ, ω) give rise to a Hom-Lie 2-algebra morphism $f = (f_0, f_1, f_2)$ from \mathfrak{g} to the derivation Hom-Lie 2-algebra $\text{DER}(\mathfrak{h})$ given in Theorem 3.8, where f_0, f_1, f_2 are given by

$$f_0(x) = \rho_x, \quad f_1 = 0, \quad f_2(x, y) = -\omega(x, y), \quad \forall x, y \in \mathfrak{g}.$$

Conversely, for any morphism $f = (f_0, f_1, f_2)$ from \mathfrak{g} to $\text{DER}(\mathfrak{h})$, there is a diagonal non-abelian extension $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\rho, \omega)}, \phi)$ of \mathfrak{g} by \mathfrak{h} , where ρ and ω are given by

$$\rho_x = f_0(x), \quad \omega(x, y) = -f_2(x, y), \quad \forall x, y \in \mathfrak{g}.$$

Proof. Let $\hat{\mathfrak{g}} = (\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\rho, \omega)}, \phi)$ be a diagonal non-abelian extension of \mathfrak{g} by \mathfrak{h} . By (23), $\rho_x \in \text{Der}(\mathfrak{h})$ for all $x \in \mathfrak{g}$. By (21) and (22), we have

$$\begin{aligned} f_0(\phi_{\mathfrak{g}}(x)) &= \rho_{\phi_{\mathfrak{g}}(x)} = \phi_{\mathfrak{h}} \circ \rho_x \circ \phi_{\mathfrak{h}}^{-1} = \text{Ad}_{\phi_{\mathfrak{h}}}(\rho_x) = \text{Ad}_{\phi_{\mathfrak{h}}}(f_0(x)), \\ f_2(\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y)) &= -\omega(\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y)) = \phi_{\mathfrak{h}}(-\omega(x, y)) = \phi_{\mathfrak{h}}(f_2(x, y)). \end{aligned}$$

By (24), we have

$$df_2(x, y) = \text{ad}_{-\omega(x, y)} = \rho_{[x, y]_{\mathfrak{g}}} - [\rho_x, \rho_y]_{\phi_{\mathfrak{h}}} = f_0([x, y]_{\mathfrak{g}}) - [f_0(x), f_0(y)]_{\phi_{\mathfrak{h}}}.$$

By (25), we have

$$\begin{aligned} l'_2(f_0(\phi_{\mathfrak{g}}(x)), f_2(y, z)) + c.p. &= \rho_{\phi_{\mathfrak{g}}(x)}(-\omega(y, z)) + c.p. \\ &= -\omega([x, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)) + c.p. \\ &= f_2([x, y]_{\mathfrak{g}}, \phi_{\mathfrak{g}}(z)) + c.p.. \end{aligned}$$

Thus, (f_0, f_1, f_2) is a morphism from \mathfrak{g} to the derivation Hom-Lie 2-algebra $\text{DER}(\mathfrak{h})$.

The converse part is easy to be checked. The proof is finished. ■

Theorem 4.9. The isomorphism classes of diagonal non-abelian extensions of \mathfrak{g} by \mathfrak{h} are in one-to-one correspondence with the homotopy classes of Hom-Lie 2-algebra morphisms from \mathfrak{g} to the derivation Hom-Lie 2-algebra $\text{DER}(\mathfrak{h})$.

Proof. Let $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\omega^1, \rho^1)}, \phi)$ and $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(\omega^2, \rho^2)}, \phi)$ be diagonal non-abelian extensions of \mathfrak{g} by \mathfrak{h} . By Theorem 4.8, we have two Hom-Lie 2-algebra morphisms from \mathfrak{g} to the strict Hom-Lie 2-algebra $\text{DER}(\mathfrak{h})$, given by

$$g = (g_0 = \rho^1, g_1 = 0, g_2 = -\omega^1), \quad f = (f_0 = \rho^2, f_1 = 0, f_2 = -\omega^2).$$

Assume that the two extensions are isomorphic. Then there is a Hom-Lie algebra morphism $\theta : \mathfrak{g} \oplus \mathfrak{h} \rightarrow \mathfrak{g} \oplus \mathfrak{h}$, such that we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{h} & \xrightarrow{\iota} & \mathfrak{g} \oplus \mathfrak{h}_{(\rho^2, \omega^2)} & \xrightarrow{\text{pr}} & \mathfrak{g} \longrightarrow 0 \\ & & \parallel & & \theta \downarrow & & \parallel \\ 0 & \longrightarrow & \mathfrak{h} & \xrightarrow{\iota} & \mathfrak{g} \oplus \mathfrak{h}_{(\rho^1, \omega^1)} & \xrightarrow{\text{pr}} & \mathfrak{g} \longrightarrow 0, \end{array}$$

where pr is the projection. Since for all $x \in \mathfrak{g}$, $\text{pr}(\theta(x)) = x$, we can assume that $\theta(x + u) = x - \varphi_{\theta}(x) + u$ for some linear map $\varphi_{\theta} : \mathfrak{g} \rightarrow \mathfrak{h}$. By $\theta \circ \phi = \phi \circ \theta$, we get

$$\phi_{\mathfrak{h}} \circ \varphi_{\theta} = \varphi_{\theta} \circ \phi_{\mathfrak{g}}. \quad (26)$$

By $\theta[x + u, y + v]_{(\rho^2, \omega^2)} = [\theta(x + u), \theta(y + v)]_{(\rho^1, \omega^1)}$, we get

$$\rho_x^1 - \rho_x^2 = \text{ad}_{\varphi_\theta(x)}, \quad (27)$$

$$\omega^1(x, y) - \omega^2(x, y) = \rho_x^2(\varphi_\theta(y)) - \rho_y^2(\varphi_\theta(x)) + [\varphi_\theta(x), \varphi_\theta(y)]_{\mathfrak{h}} - \varphi_\theta([x, y]_{\mathfrak{g}}). \quad (28)$$

We define the chain homotopy $\tau : f \Rightarrow g$ by $\tau(x) = \varphi_\theta(x)$. Since

$$g_0(x) - f_0(x) = \rho_x^1 - \rho_x^2 = \text{ad}_{\varphi_\theta(x)} = d(\tau(x)),$$

the above definition is well-defined. We go on to prove that τ is a 2-morphism from f to g . For all $x \in \mathfrak{g}$, obviously, we have

$$\phi_{\mathfrak{h}}(\tau(x)) = \phi_{\mathfrak{h}}(\varphi_\theta(x)) = \varphi_\theta(\phi_{\mathfrak{g}}(x)) = \tau(\phi_{\mathfrak{g}}(x)).$$

By (28), we have

$$\begin{aligned} f_2(x, y) - g_2(x, y) &= (-\omega^2(x, y)) - (-\omega^1(x, y)) = \omega^1(x, y) - \omega^2(x, y) \\ &= \rho_x^2(\varphi_\theta(y)) - \rho_y^2(\varphi_\theta(x)) + [\varphi_\theta(x), \varphi_\theta(y)]_{\mathfrak{h}} - \varphi_\theta([x, y]_{\mathfrak{g}}) \\ &= l_2(\rho_x^2, \varphi_\theta(y)) - l_2(\rho_y^2, \varphi_\theta(x)) + l_2(\text{ad}_{\tau(x)}, \tau(y)) - \tau([x, y]_{\mathfrak{g}}) \\ &= l_2(f_0(x), \tau(y)) - l_2(f_0(y), \tau(x)) + l_2(d(\tau(x)), \tau(y)) - \tau([x, y]_{\mathfrak{g}}) \\ &= l_2(f_0(x) + d(\tau(x)), \tau(y)) + l_2(\tau(x), f_0(y)) - \tau([x, y]_{\mathfrak{g}}) \\ &= l_2(g_0(x), \tau(y)) + l_2(\tau(x), f_0(y)) - \tau([x, y]_{\mathfrak{g}}). \end{aligned}$$

Thus, τ is a 2-morphism from f to g , which implies that f and g are homotopic Hom-Lie 2-algebra morphisms. Therefore, isomorphic diagonal non-abelian extensions of \mathfrak{g} by \mathfrak{h} correspond to homotopic Hom-Lie 2-algebra morphisms from \mathfrak{g} to the derivation Hom-Lie 2-algebra $\text{DER}(\mathfrak{h})$.

Conversely, let $g = (g_0, g_1 = 0, g_2)$ and $f = (f_0, f_1 = 0, f_2)$ be homotopic morphisms from \mathfrak{g} to the derivation Hom-Lie 2-algebra $\text{DER}(\mathfrak{h})$. By Theorem 4.8, we have two diagonal non-abelian extensions $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(g_0, -g_2)}, \phi)$ and $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(f_0, -f_2)}, \phi)$ of \mathfrak{g} by \mathfrak{h} . Assume that τ is a 2-morphism from f to g . We define $\theta : \mathfrak{g} \oplus \mathfrak{h} \rightarrow \mathfrak{g} \oplus \mathfrak{h}$ by

$$\theta(x + u) = x - \tau(x) + u.$$

It is straightforward that θ is a Hom-Lie algebra morphism making the diagram in Definition 4.2 commutative. Therefore, $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(g_0, -g_2)}, \phi)$ and $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{(f_0, -f_2)}, \phi)$ are isomorphic diagonal extensions of \mathfrak{g} by \mathfrak{h} . Thus, homotopic Hom-Lie 2-algebra morphisms from \mathfrak{g} to the derivation Hom-Lie 2-algebra $\text{DER}(\mathfrak{h})$ correspond to isomorphic diagonal non-abelian extensions of \mathfrak{g} by \mathfrak{h} . This finishes the proof. ■

5 Outlook

There are several ways to study non-abelian extensions of Lie algebras. The authors classified non-abelian extensions of Lie algebras using the second non-abelian cohomology group [12]. One can also use outer derivations to study non-abelian extensions of Lie algebras and characterize the existence of a non-abelian extension using a cohomological class [1, 2, 10]. Furthermore, one can also use Maurer-Cartan elements to describe non-abelian extensions [8]. In the sequel, we give a sketch on how to generalize these ways to study diagonal non-abelian extensions of Hom-Lie

algebras. In all, we think that the most nontrivial part is defining the notion of a derivation of a Hom-Lie algebra and constructing the associated derivation Hom-Lie 2-algebra. Thus, we only study diagonal non-abelian extensions of Hom-Lie algebras using the derivation Hom-Lie 2-algebra in detail in this paper and omit details of other ways.

For non-abelian extensions of Hom-Lie algebras, we can define a non-abelian 2-cocycle using Eqs. (21)-(25). Namely, a pair (ρ, ω) is called a nonabelian 2-cocycle if Eqs. (21)-(25) are satisfied. We can further define an equivalence relation using Eqs. (26)-(28), i.e. two nonabelian 2-cocycles (ρ^1, ω^1) and (ρ^2, ω^2) are equivalent if Eqs. (26)-(28) are satisfied. Then we obtain the second nonabelian cohomology group, by which we can classify diagonal non-abelian extensions of Hom-Lie algebras.

By (22)-(24), we deduce that the pair (ρ, ω) give rise to a Hom-Lie algebra morphism $\bar{\rho} : \mathfrak{g} \longrightarrow \text{Out}(\mathfrak{h})$, where $\bar{\rho} = \pi \circ \rho$ and π is the quotient map in the following exact sequence:

$$0 \longrightarrow \text{Inn}(\mathfrak{h}) \longrightarrow \text{Der}(\mathfrak{h}) \xrightarrow{\pi} \text{Out}(\mathfrak{h}) \longrightarrow 0. \quad (29)$$

Note that $\text{Der}(\mathfrak{h})$ is a non-abelian extension of $\text{Out}(\mathfrak{h})$ by $\text{Inn}(\mathfrak{h})$. We have another abelian extension of Hom-Lie algebras:

$$0 \longrightarrow \text{Cen}(\mathfrak{h}) \longrightarrow \mathfrak{h} \xrightarrow{\text{ad}} \text{Inn}(\mathfrak{h}) \longrightarrow 0. \quad (30)$$

Under the assumption that both (29) and (30) are diagonal non-abelian extensions, for any Hom-Lie algebra morphism $\bar{\rho} : \mathfrak{g} \longrightarrow \text{Out}(\mathfrak{h})$, the existence of a diagonal non-abelian extension that inducing the morphism $\bar{\rho}$ can be characterized by a condition on a cohomological class in $\mathcal{H}^3(\mathfrak{g}, \text{Cen}(\mathfrak{h}))$.

For a differential graded Hom-Lie algebra (DGH LA for short) $(L, [\cdot, \cdot]_L, \phi_L, d)$, the set $MC(L)$ of Maurer-Cartan elements is defined by

$$MC(L) \triangleq \{\alpha \in L^1 \mid d\alpha + \frac{1}{2}[\alpha, \alpha]_L = 0, \phi_L \alpha = \alpha\}.$$

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ be a Hom-Lie algebra. Then $(C(\mathfrak{g}, \mathfrak{g}) = \oplus_k C^k(\mathfrak{g}, \mathfrak{g}), [\cdot, \cdot]_{\phi_{\mathfrak{g}}}, \text{Ad}_{\phi_{\mathfrak{g}}})$ is a graded Hom-Lie algebra, where $[\cdot, \cdot]_{\phi_{\mathfrak{g}}}$ and $\text{Ad}_{\phi_{\mathfrak{g}}}$ are given by

$$[P, Q]_{\phi_{\mathfrak{g}}} = P \circ Q - (-1)^{pq} Q \circ P, \quad \forall P \in C^{p+1}(\mathfrak{g}, \mathfrak{g}), Q \in C^{q+1}(\mathfrak{g}, \mathfrak{g}), \quad (31)$$

$$(\text{Ad}_{\phi_{\mathfrak{g}}} P)(x_1, \dots, x_{p+1}) = \phi_{\mathfrak{g}}(P(\phi_{\mathfrak{g}}^{-1} x_1, \dots, \phi_{\mathfrak{g}}^{-1} x_{p+1})), \quad \forall P \in C^{p+1}(\mathfrak{g}, \mathfrak{g}), \quad (32)$$

in which $C^k(\mathfrak{g}, \mathfrak{g}) = \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{g})$ and $P \circ Q \in C^{p+q+1}(\mathfrak{g}, \mathfrak{g})$ is defined by

$$P \circ Q(x_1, \dots, x_{p+q+1}) = \sum_{\sigma \in (q+1, p) - \text{unshuffles}} (-1)^{\sigma} \quad (33)$$

$$\phi_{\mathfrak{g}} P \left(\phi_{\mathfrak{g}}^{-1} Q(\phi_{\mathfrak{g}}^{-1} x_{\sigma(1)}, \dots, \phi_{\mathfrak{g}}^{-1} x_{\sigma(q+1)}), \phi_{\mathfrak{g}}^{-1} x_{\sigma(q+2)}, \dots, \phi_{\mathfrak{g}}^{-1} x_{\sigma(p+q+1)} \right). \quad (34)$$

Furthermore $(C(\mathfrak{g}, \mathfrak{g}), [\cdot, \cdot]_{\phi_{\mathfrak{g}}}, \text{Ad}_{\phi_{\mathfrak{g}}}, \partial)$ is a DGH LA, where $\partial P = (-1)^{k+1} [\mu_{\mathfrak{g}}, P]_{\phi_{\mathfrak{g}}}$ for all $P \in C^k(\mathfrak{g}, \mathfrak{g})$ and $\mu_{\mathfrak{g}}$ is the Hom-Lie bracket on \mathfrak{g} , i.e. $\mu_{\mathfrak{g}}(x, y) = [x, y]_{\mathfrak{g}}$. See [7] for more details.

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \phi_{\mathfrak{h}})$ be two Hom-Lie algebras. Let $\mathfrak{g} \oplus \mathfrak{h}$ be the Hom-Lie algebra direct sum of \mathfrak{g} and \mathfrak{h} , where the bracket is defined by $[x + u, y + v] = [x, y]_{\mathfrak{g}} + [u, v]_{\mathfrak{h}}$, and the algebra morphism ϕ is defined by $\phi(x + u) = \phi_{\mathfrak{g}}(x) + \phi_{\mathfrak{h}}(u)$. Then there is a DGH LA $(C(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{g} \oplus \mathfrak{h}), [\cdot, \cdot], \text{Ad}_{\phi}, \partial)$, where $\partial P = (-1)^{k+1} [\mu_{\mathfrak{g}} + \mu_{\mathfrak{h}}, P]_{\phi}$ for all $P \in C^k(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{g} \oplus \mathfrak{h})$. Define $C_{>}^k(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h}) \subset C^k(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h})$ by

$$C^k(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h}) = C_{>}^k(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h}) \oplus C^k(\mathfrak{h}, \mathfrak{h}).$$

Denote by $C_{>}(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h}) = \oplus_k C_{>}^k(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h})$. It is straightforward to see that $(C_{>}(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h}), [\cdot, \cdot]_{\phi}, \text{Ad}_{\phi}, \partial)$ is a sub-DGHLA of $(C(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{g} \oplus \mathfrak{h}), [\cdot, \cdot]_{\phi}, \text{Ad}_{\phi}, \partial)$. We denote by $(L, [\cdot, \cdot]_{\phi}, \text{Ad}_{\phi}, \partial)$ this sub-DGHLA, where $L^k = C_{>}^{k+1}(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h})$. Obviously, its degree 0 part $C_{>}^0(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h}) = \text{Hom}(\mathfrak{g}, \mathfrak{h})$ is abelian. Then it is straightforward to check that (ρ, ω) satisfy Eqs. (21)-(25), i.e. (ρ, ω) give rise to a diagonal non-abelian extension if and only if $\rho + \omega$ is a Maurer-Cartan element in the DGHLA $(L, [\cdot, \cdot]_{\phi}, \text{Ad}_{\phi}, \partial)$.

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